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# COMPLEXITY DYNAMICS OF EVOLUTION OF MATURE POPULATION

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## ABSTRACT

All the biological systems in nature exhibit enormous diversity during evolution. The dynamics of such systems are very interesting. Studying such dynamics is a cumbersome job but it enables humans to find the reasons of extinction of certain species or unbalance in such systems. These studies can help in finding some techniques to control the existing chaos, which could provide solution to many existing problems. In this paper, we have worked with a single-species model with stage structure for the dynamics in a wild animal population for which births occur in a single pulse once per time period. We have tried to analyze regularity and chaos in the system by finding bifurcations, Lyapunov exponents, topological entropy of the evolving system and the correlation dimension of the chaotic attractor. Finally, the results and findings are briefly discussed in the last section.

## 1. Introduction

Usually biological systems are complex and multicomponent, [1]. They are spatially structured and their individual elements possess individual properties. Such complexity also effects the system significantly during evolution. Natural processes tend to vary over time and space, as well as between species. In recent years there has been a great emphasis on three concerning phrases: nonlinear dynamics, chaos, and complexity. This interest has led to a large number of popular-science articles covering models and graphics to explain chaos, regularity and chaos control in certain cases. Henri Poincaré (1854–1912), a late-nineteenth century French mathematician was the first one to extensively study topology and dynamic systems. All natural systems exhibit massive diversity in behavior during evolution. Complex systems are characterized by an internal structure which is built by numerous and varied processes, subsystems and interconnections. Systems featured by complexity display a number of properties such as uncertainty, interactions at different levels, self-organization and nonlinear feedback. Due to its nonlinear structure, such systems may display the properties like complexity and chaos. Elaborate descriptions on complexity can viewed from some well-written articles [2 – 7].

A chaotic system can better understood by measuring Lyapunov exponents (LCEs), topological entropies and correlation dimension. Positive LCE during evolution signifies chaotic evolution. Topological entropy, a non-negative number, provides a perfect way to measure complexity of a dynamical system. For a system, more topological entropy signifies more complexity. Actually, it measures the evolution of distinguishable orbits over time, thereby providing an idea of how complex the orbit structure of a system is [8–16]. It describes the rate of mixing of a dynamical system. It related to Lyapunov exponents both through the dependence of rate and to the ergodicity. For a system having non-zero topological entropy, the rate of mixing must be exponential which is comparable to Lyapunov exponents. However, such exponentiality is not relative to time, rather to the number of discrete steps through which the system has evolved. Positivity of Lyapunov exponent and topological entropy are characteristic of chaos. LCE's provide the rate of divergence of orbits, which initially started very closely. The book by Nagashima and Baba, [16]. gives a very clear definition of topological entropy.

While working with population dynamics in many models the increases in population due to birth assumed time-independent, but that is not always the case. In many cases, some species reproduce only during a single period of the year. We work here with a single-species model with stage structure for the dynamics in a wild animal population for which births occur in a single pulse once per time period. This model studied here proposed by Tang and

Cheng, [17]. We obtained bifurcations, LCE's and topological entropy to analyze and measure chaos and complexities in the system. The sequence of bifurcations, leading to chaos shows that the dynamical behaviors of the single species model with birth pulses are very complex and chaotic.

2. The Model

The change in population size, in absence of stage structure, assumed to happen as per the population growth ratio, [17], represented by the equation

$N = B(N)N - d N$ (1)

Here, $d > 0$ is the death rate constant, and $B(N)$ is the birth rate function satisfying some basic assumptions for $N \in (0, \infty)$ as:

(i) $B(N) > 0$ ;

(ii) $B(N)$ is continuously differentiable with $B'(N) < 0$ ;

(iii) $B(0^+) > d > B(\infty)$

The condition (iii) implies the existence of a carrying capacity K of the environment such that

$B(N) > d$ for $N < K$, and $B(N) < d$ for $N > K$.

This implies an unique steady state equilibrium $N^*$ exists of equation (1) such that when $N^* = K$ , $B(N^*) = d$. Two examples of birth functions B(N) usually found in biological literatures that satisfy above conditions (i) – (iii) are obtained as:

(a) $B_1(N) = b\, e^{-N}$, with $b > d$ ;

(b) $B_2(N) = \frac{p}{q + N^n}$ , with $p, q, n > 0$ and $\frac{p}{q} > d$ .

The function $B_1(N)$ is known as the Ricker function and the function $B_2(N)$ is known as the Beverton–Holt function.

We assume now that the single species population in model (2.1) has stage structure, and that the population N is divided into immature and mature classes, with the size of each class given by x(t) and y(t), respectively, so that N (t) = x(t) + y(t), and only the mature population can reproduce.

With the assumption that the single species population in model(1) has stage structure and that the population N is divided into immature and mature classes, with the size of each class given by x(t)and y(t), respectively, so that N(t)= x(t)+y(t), and only the mature population can reproduce, [17].

Taking Ricker function, $B(N) = e^{-(x+y)}$, [17], a discrete form dynamic model derived leading to the following two equations:

(2)

$$x_{n+1} = x_n\, e^{-(\delta+d)} + b\,[y + x_n\,(1 - e^{-\delta})]\, e^{-[d + e^{-\delta}(x_n + y_n]}$$

$$x_{n+1} = e^{-\delta}(1 - e^{-\delta})x_n + e^{-d}\, y$$

## 3. Numerical Simulations:

To investigate the dynamics of evolution of system (2), various numerical simulations performed such as obtaining bifurcation diagrams, calculating Lyapunov exponents, topological entropy and correlation dimensions of the system for different cases. Explanations of these are as follows:

(a) **Bifurcations:**

Bifurcations scenario in any nonlinear dynamical system play a very important role in studying the evolutionary properties of the system. From bifurcations, one observes the qualitative changes in the evolving system in various parameter space. Here, varying parameter b, and keeping d= 0.7 and δ = 0.5, bifurcations of system (2) are obtained. Bifurcation figures shown in Fig. 1, are for three ranges of b; 0 □ b □ 600, 300 □ b □ 350 and for 550 □ b □ 600. One observes, as parameter b increases, bifurcation phenomena from one cycle to two cycle, then 4 cycle, then 8 cycle etc., a period doubling criteria and finally leading to chaos. Within chaotic region one clearly also observes various periodic windows. The appearance of periodic windows within chaotic region of bifurcations is an indication of intermittency and other complex phenomena. Periodic windows become gradually shorter and appearance become more frequent while moving forward in parameter space.

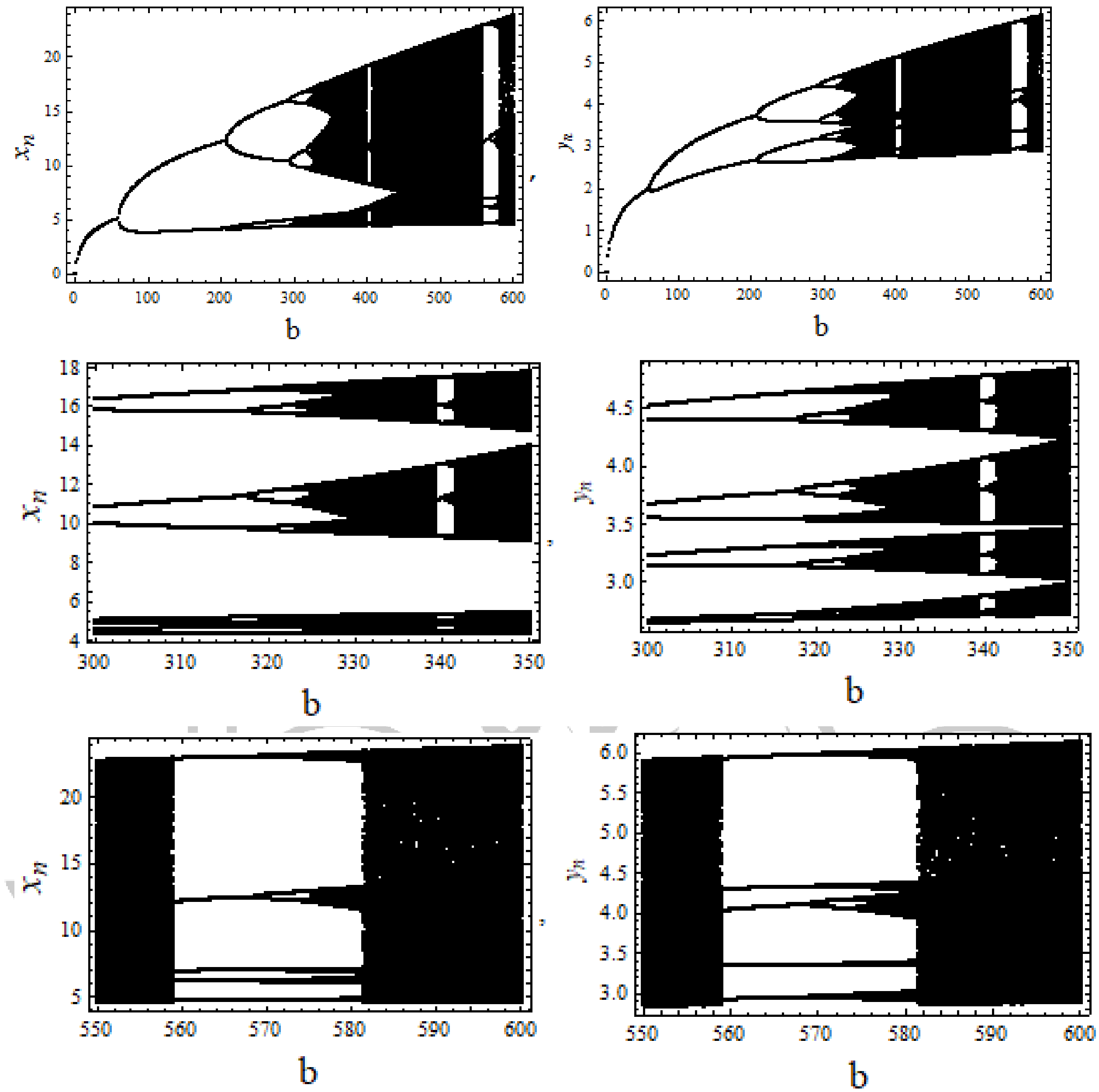


Fig.1: Bifurcation diagrams along x- and y- axes of system (2) for 0 □ b □ 600, 300 □ b □ 350 and for 550 □ b □ 600 when d= 0.7 and δ = 0.5.

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**a) Regular Periodic and chaotic attractors:**

With fixed values d= 0.7 and δ = 0.5 and changing values of parameter b, different periodic attractors are obtained, Fig. 2, for showing regular motion of system (2). However, when values of b increased substantially, system evolution changes from regularity to chaos. These can observed from time series plots and plots of chaotic attractors shown in Fig. 3.

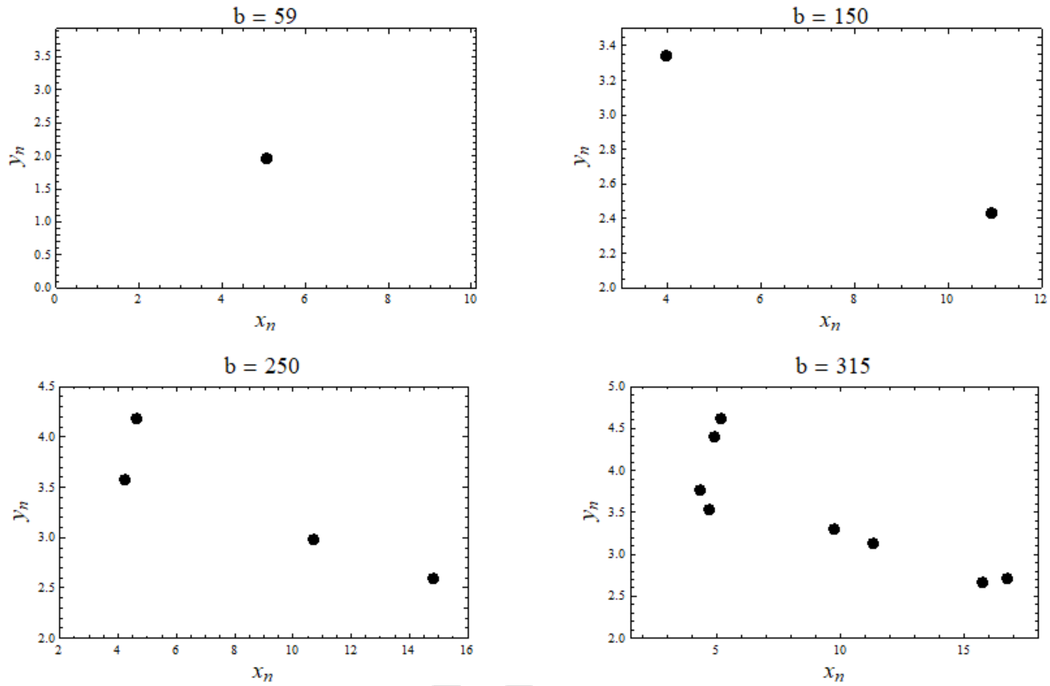


Fig. 2: Periodic attractors of system (2) with periods 1, 2, 4, 8 for different values of parameter b. Other parameters are d= 0.7 and δ = 0.5.

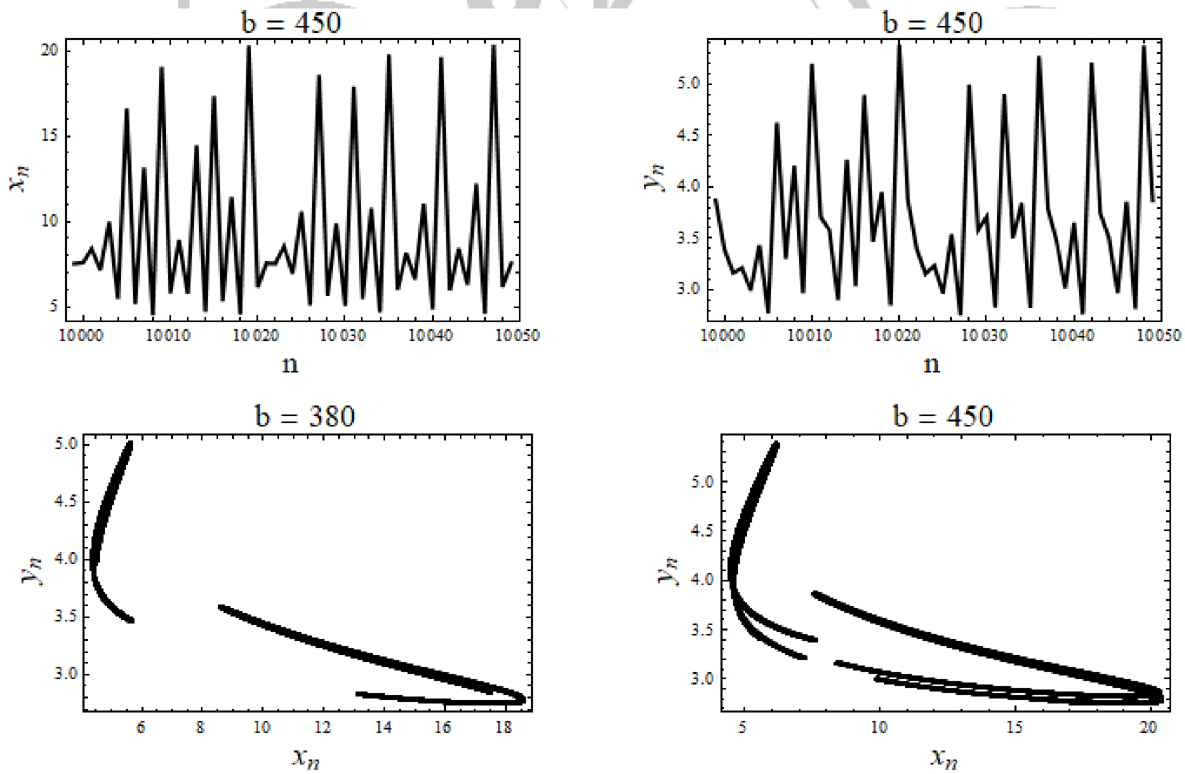


Fig. 3: Chaotic time series plots, (upper row), and plots of chaotic attractors, (lower row) are shown here. The other parameters are taken as d= 0.7 and δ = 0.5.

**(c) Lyapunov Exponents (LCE's) :**

Lyapunov exponents are considered generalizations of the eigenvalues of steady-state and limit-cycle solutions to differential equations. The eigenvalues of a limit cycle characterize the rate at which nearby trajectories converge or diverge from the cycle. The Lyapunov exponents do the same thing, but for arbitrary trajectories, not just the special ones that are periodic. Calculation of Lyapunov exponents for nonlinear systems involves numerical integration of the underlying differential equations of motion, together with their associated equations of variation. Actually, a Lyapunov exponent measures how "complex" the map is. The system evolutions considered, respectively, regular or chaotic whether Lyapunov exponents are < 0 or > 0. Plots of Lyapunov exponents for parameters d = 0.7, b = 540 and δ = 0.5 for two different ranges of iterations are shown in Fig. 4.

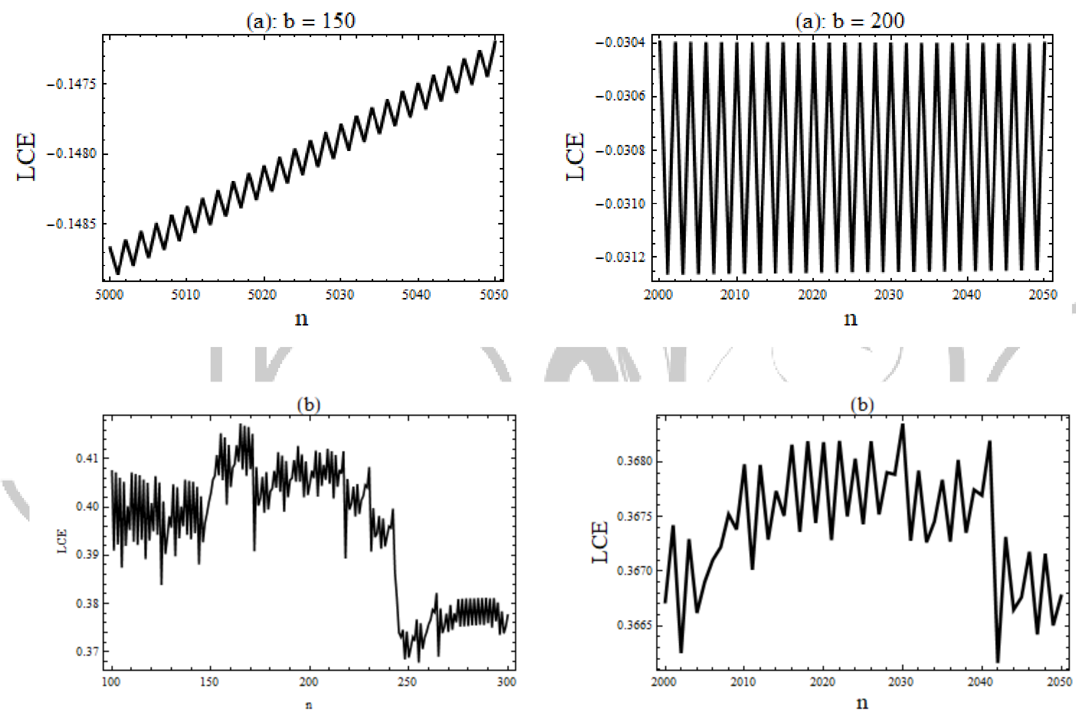


Fig. 4: Plots of LCEs for d= 0.7, δ = 0.5 then, (i) upper row for regular cases for different values of b and (ii) lower row for chaotic case for same b = 540 but for different ranges of iterations.

(b) **Topological Entropy:**

The topological entropy measures the growth of the number of periodic points. .In communication theory, a measure of uncertainty or randomness that is related to information. The greater the entropy, the greater the uncertainty, and the greater the amount of information capable of being transmitted. However, once received, information represents a decrease in uncertainty. The flip of a fair coin yields one bit of entropy with 0 or 1 representing heads or tails. A binary bit with equiprobability of a 0 or 1 is random. Also, the higher the probability of an event or a state, the lower the entropy. Back in 1948, Claude Shannon helped established the theoretical basis for the development of information and communication theory with his equation of entropy.

A second intuitive interpretation of entropy is as a measure of the disorder in a system. There are interesting examples of systems that appear to develop more order as their entropy (and temperature) rises. These are systems where adding order of one, visible type (say, crystalline or orientation ally order) allows increased disorder of another type (say, vibrational disorder). Entropy is a precise measure of disorder but is not the only possible or useful measure. The topological entropy is a nonnegative number which measures the complexity of the system. Roughly, it measures the exponential growth rate of the number of distinguishable orbits as time advances. To name a periodic orbit, we need only choose one of its cyclic permutations. The number of distinct periodic orbits grows rapidly with the length of the period. A simple indicator of the complexity of a dynamical system is its topological entropy. In the one-dimensional setting, the topological entropy, which we denote by, is a measure of the growth of the number of periodic cycles as a function of the symbol string length (period).

Measure of theoretic entropy, which is also called the Kolmogorov—Sinai invariant, was defined for measure preserving transformations of probability measure spaces. Adler, Konhelm and McAndrew in the 1960s, [8], first introduced the concept of topological entropy. Topological entropy describes the rate of mixing of a dynamical system. It is related to Lyapunov exponents both through the dependence of rate and to the ergodicity.

For a system having non-zero topological entropy, the rate of mixing must be exponential which is comparable to Lyapunov exponents. Though such exponentiality is not relative to time, rather to the number of discrete steps through which the system has evolved. Positivity of Lyapunov exponent and topological entropy are characteristic of chaos. The book by Nagashima and Baba, [16], gives a very clear definition of entropy. The graphics for Topological Entropy are shown below:

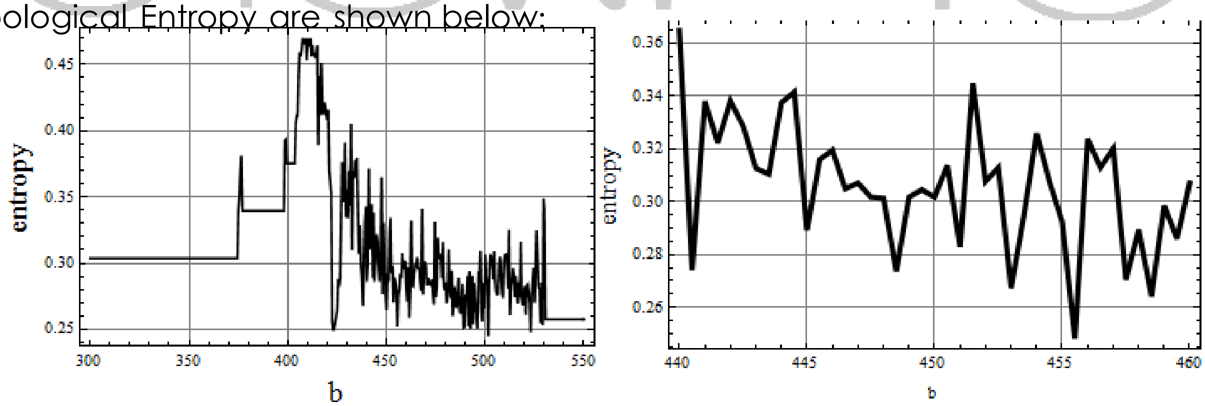


Fig. 5: Plots of topological entropies are presented with d= 0.7 and $\delta = 0.5$ and 300 □ b□ 550 and 440 □ b□ 460.

(c) **Correlation Dimension**:

Correlation dimension provides the dimensionality of the evolving chaotic attractor. It is a kind of fractal dimension and its numerical value is always non-integer. Being one of the characteristic invariants of

nonlinear system dynamics, the correlation dimension actually gives a measure of complexity for the underlying attractor of the system. The procedure to calculate correlation dimension is statistical and that we have followed here Martelli, [18].

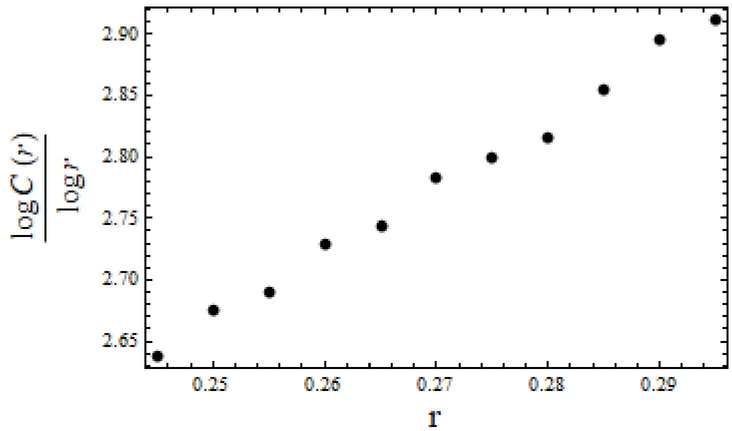


Fig. 6: Plot of correlation integral curve for d= 0.7, b = 540 and δ = 0.5

Using linear fit to the correlation data used to obtain correlation curve, Fig. 6, one obtains

Y = 5.41035 x + 1.31561

Thus, the correlation dimension obtained approximately as DC  1.31. In a similar way correlation dimension of every chaotic attractor, emerging during evolution can obtained.

4. **Discussions**:

Complexity and chaotic evolutionary motion have been discussed for discrete mature population model proposed in [17]. Bifurcation diagrams, Fig. 1, show the system evolve through period doubling root to chaos. Measures of complexity; such as Lyapunov exponents, topological entropies, correlation dimension have been calculated and shown through figures, Fig. 2 – Fig. 7. Plots of LCEs and topological entropies show clearly the complexity nature of the system. The correlation dimension for the chaotic attractor when parameters d= 0.7, b = 540 and δ = 0.5, be obtained, approximately as, DC  2.45.

References:

1. R. M. May (1976): Simple mathematical models with very complicated dynamics. Nature, 261, pages459–467.

2. W. Weaver,(1948): "Science and complexity," American Scientist, vol. 36, no. 4, p. 536, .

3. H. A. Simon,(1962): "The architecture of complexity," Proceedings of the American Philosophical Society, vol. 106(6): 467–482.

4. Gleick, J. (1987). "Chaos: Making a New Science". A Cardinal Book, Sphere Books Ltd., London.

5. Keith, W.,Cynthia, F.,Calvin, L.S.(1998). "New Directions in Systems Theory": Chaos and Complexity. *Social Work,* Vol. 43, no.4, p. 357–372,

6. Manson, S. M. (2001). Simplifying complexity: a review of complexity theory. Geoforum, 32(3): 405-414.

7. Yaneer Bar-Yam ( 1992): Dynamics of Complex Systems. Addison-Wesley, Reading, Massachusetts.

8. Adler, R L Konheim, A G McAndrew, M H. (1965): Topological entropy, Trans. Amer. Math. Soc. , 114: 309-319

9. Andrecut, M. and Kauffman, S. A. (2007): Chaos in a Discrete Model of a Two-Gene System.Physics Letters, A 367: 281-287.

10. R. Bowen, R . Topological entropy for noncompact sets (1973): Trans. Amer. Math. Soc. 1973: 184: 125- 136.

11. K. Butt (2014): An introduction to Topological entropy. https://math.uchicago.edu/~may/REU2014/REUPapers/Butt.pdf

12. Benettin, G.; Galgani, L.; Giorgilli, A.; Strelcyn, J. M. (1980). "Lyapunov characteristic exponents for smooth dynamical systems and for hamiltonian systems; a method for computing all of them. Part 1 & II: Theory".Meccanica 15: 9 – 20 & 21 – 30.

13. Abarbanel, H.D.I.; Brown, R.; Kennel, M.B. (1992). "Local Lyapunov exponents computed from observed data". *Journal of Nonlinear Science* 2 (3), pp 343-365.

14. Gribble, S. 1995, Topological Entropy as a Practical Tool for Identification and Characterization of Chaotic System. Physics 449 Thesis.

15. Balmforth, N. J., Spiegel, E. A. and Tresser, C. 1994, Topological entropy of one dimensional maps: Approximations and bounds. Phys. Rev. Lett., 72, 80 – 83.

16. Nagashima, H. and Baba, Y. 200, Introduction to Chaos: Physics and Mathematics of Chaotic Phenomena. Overseas Press India Private Limited.

17. Tang S., Chen L. (2002): Density-dependent birth rate, birth pulses and their population dynamic consequences. J. Math. Biology. 44 (2):1 – 15.

18. M. Martelli (1999), Introduction to Discrete Dynamical Systems and Chaos, John Wiley & Sons, Inc, New York.